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DISPERSION OF NONLINEAR SPATIAL WAVES IN A ROD*

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A nonlinear generalization of the dispersion equations of three-dimensional oscillations is constructed for an inextensible rod. The dispersion equation is obtained for waves of any amplitude in the closed form,containing a solution of a certain system of two transcendental equations. For small amplitudes the final relationships are given in explicit form, taking into account the nonlinearity in the first approximation. The dispersion relation contains, just as in the linear case, the mean tensile force, the wave number and frequency, as well as the parameters defining the oscillation amplitudes. From the dispersion relation it follows that the larger the oscillation amplitude, the greater the wave velocity. The interaction of the transverse oscillations at different planes, absent in the linear case, is described, the characteristics of the "effective" rod computed and the corresponding averaged equations constructed. A nonlinear generalization of the dispersion equation was constructed earlier in /1/ for the plane oscillations of a compressible rod, for the case of small finite amplitudes. An analogous problem was studied in /2/ for the oscillations of an inextensible rod for the case of finite amplitudes, and in addition a concept of an "effective" rod was introduced, its characteristics computed, and the corresponding averaged equations constructed. The present paper generalizes the results of /2/.

1. Formulation of the problem. We consider, using the Cartesian coordinate x^1, x^2, x^3 system, an infinite homogeneous inextensible rod rectilinear in the undeformed state. The elastic line of the rod is defined by the equations (ξ is the rod axis arc length, and t is time)

$$x^i = r^i(\xi, t), \quad i = 1, 2, 3$$

Let the inertia tensor of the transverse cross section of the rod $I^{\alpha\beta}$ be spherical (S is the area of transverse cross section and $\delta^{\alpha\beta}$ is the Kronecker delta)

$$I^{\alpha\beta} = h^2 S \delta^{\alpha\beta}$$

We shall regard the above equation as the definition of h which is of the order of the transverse cross section diameter. The Lagrangian of the rod relative to the area of transverse cross section S and Young's modulus E, has the form

$$2\Lambda = h^2 r^i_{\xi\xi} r_{i\xi\xi} - c_0^{-r} r^i_t r_{it} \quad (c_0{}^2 = E/\rho) \tag{1.1}$$

Here ρ is density and the subscripts ξ and t denote differentiation with respect to ξ and t. The first term in (1.1) is the energy of the flexure and the second term denotes the kinetic energy of unit rod length. The condition of inextensibility has the form

$$r^i_\xi r_{i\xi} = 1 \tag{1.2}$$

We shall consider the motions of a special type of a rod, namely, let the functions $r^i(\xi, t)$ be represented in the form

$$r^i = v^i(\xi, t) + \psi^i(\theta, \xi, t) \tag{1.3}$$

where θ is a function of ξ and t, and ψ^i are 2π-periodic functions of θ. The above conditions do not restrict the generality of the argument. The assumptions concerning the character of the dependence of ${}^iv^i, \psi, \theta$ in ξ, t introduce certain restrictions, and are as follows. Let the characteristic scales L and T of variation of the functions $\theta_t, \theta_\xi, v_t{}^i, v_\xi{}^i$ and ψ^i in ξ and t at constant θ be much larger than the characteristic scales l and τ of variation of the functions $\psi^i(\theta(t, \xi), t, \xi)$ in ξ and t, respectively. In this manner we study the motions of a rod during which the rapid oscillations ψ^i, the characteristics of which oscillate slowly, are superimposed on the slow, smooth "background" v^i. The description of such motions is helped by the presence of a small parameter $\delta = \max\{l/L, \tau/T\}$. In the zero approximation with respect

*Prikl.Matem.Mekhan.,46,No.2,pp.324-330,1982 to this parameter the quantities θ_t, θ_ξ, v_t^i, v_ξ^i are constant and the motion represents a nonlinear propagating wave. The next higher approximation yields the equations for the modulation of these quantities, and derivation of these equations is the aim of the present work. In accordance with the Whitham method /3/ the averaged Lagrangian

$$\langle\Lambda\rangle = \frac{1}{2\pi}\int_0^{2\pi} \Lambda\, d\theta$$

must be computed from the solutions of the propagating wave type. In what follows, we shall denote by symbol $\langle\ \rangle$ the integral in θ over the interval $[0, 2\pi]$, referred to 2π. This yields a function of θ_t, θ_ξ, v_t^i, v_ξ^i, and of other slowly varying quantities related to the wave amplitudes. The Euler equations for the action defined by the averaged Lagrangian are found to represent the required equations for the modulations.

2. Averaged Lagrangian.

Substituting (1.3) into (1.1), neglecting terms small in δ and integrating with respect to θ, we obtain the following expression for the averaged Lagrangian:

$$\langle 2\Lambda\rangle = \langle h^2\theta_\xi^4 \psi_{\theta\theta}^i \psi_{i\theta\theta} - \theta_t^2 c_0^{-2} \psi_\theta^i \psi_{i\theta}\rangle - c_0^{-2} v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

The last term is proportional to δ^2, but it may also turn out to be important if e.g. the amplitude of the microoscillations is small. The condition of inextensibility (1.2) assumes the form

$$(v_\xi^i + \theta_\xi \psi_\theta^i)(v_{i\xi} + \theta_\xi \psi_{i\theta}) = 1$$

Let us make the substitution $\psi^i \to \theta, \varphi$

$$v_\xi^1 + \theta_\xi\psi_\theta^1 = \sin\Theta\cos\varphi, \quad v_\xi^2 + \theta_\xi\psi_\theta^2 = \sin\Theta\sin\varphi, \quad v_\xi^3 + \theta_\xi\psi_\theta^3 = \cos\Theta \qquad (2.1)$$

The angles Θ and φ have the sense of polar angles of the vector tangent to the rod. We also introduce the following notation:

$$2\gamma_v = v_\xi^i v_{i\xi} - 1, \quad \varkappa = h\theta_\xi, \alpha = \theta_t^2 c_0^{-2}\theta_\xi^{-2}, \quad v_\xi^3 = (1 + 2\gamma_v)^{1/2}\cos\Theta_v$$
$$v_\xi^1 = (1 + 2\gamma_v)^{1/2}\sin\Theta_v\cos\varphi_v, \quad v_\xi^2 = (1 + 2\gamma_v)^{1/2}\sin\Theta_v\sin\varphi_v$$
$$G(\Theta, \Theta_v, \varphi - \varphi_v) = \cos\Theta\cos\Theta_v + \sin\Theta\sin\Theta_v\cos(\varphi - \varphi_v)$$

The expression for the averaged Lagrangian can now be written as

$$\langle 2\Lambda\rangle = \langle \varkappa^2\Theta_\theta^2 + \varkappa^2\sin^2\Theta\varphi_\theta^2 - 2\alpha[1 + \gamma_v - (1 + 2\gamma_v)^{1/2} G(\Theta, \Theta_v, \varphi - \varphi_v)]\rangle - c_0^{-2} v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

Since conditions of periodicity were imposed on the functions ψ^i, it follows that the Lagrangian will have to be supplemented by terms containing the Lagrange multipliers. According to (2.1) the extra terms can be written in the form

$$2\alpha\lambda\langle\cos\Theta - v_\xi^3\rangle, \quad 2\alpha\mu\langle\sin\Theta\cos\varphi - v_\xi^1\rangle, \quad 2\alpha\eta\langle\sin\Theta\sin\varphi - v_\xi^2\rangle$$

Let us carry out the following substitution: instead of the Lagrange multipliers λ, μ, η we shall introduce the numbers m, Θ_*, φ_* obtained from the formulas

$$m^2 = \alpha[(v_\xi^3 + \lambda)^2 + (v_\xi^1 + \mu)^2 + (v_\xi^2 + \eta)^2]^{1/2}, \quad \cos\Theta_* = \alpha m^{-2}(v_\xi^3 + \lambda)$$
$$\sin\Theta_*\cos\varphi_* = \alpha m^{-2}(v_\xi^1 + \mu), \quad \sin\Theta_*\sin\varphi_* = \alpha m^{-2}(v_\xi^3 + \eta)$$

The averaged Lagrangian has the form

$$\langle 2\Lambda\rangle = \langle \varkappa^2\Theta_\theta^2 + \varkappa^2\sin^2\Theta\varphi_\theta^2 + 2m^2 G(\Theta, \Theta_*, \varphi - \varphi_*)\rangle -$$
$$2m^2(1 + \gamma_v)^{1/2} G(\Theta_v, \Theta_*, \varphi_v - \varphi_*) + 2\alpha\gamma_v - c_0^{-2} v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

Let us carry out the substitution $\Theta, \varphi \to \overline{\Theta}, \overline{\varphi}$ according to the formulas

$$\cos\overline{\Theta} = \cos\Theta\cos\Theta_* + \sin\Theta\sin\Theta_*\cos(\varphi - \varphi_*) = G(\Theta, \Theta_*, \varphi - \varphi_*$$
$$\sin\overline{\Theta}\cos\overline{\varphi} = \cos\Theta_*\sin\Theta\cos(\varphi - \varphi_*) - \sin\Theta_*\cos\Theta$$
$$\sin\overline{\Theta}\sin\overline{\varphi} = \sin\Theta\sin(\varphi - \varphi_*)$$

The above formulas represent in fact an orthogonal transformation from the Cartesian x, y, z-coordinates to the $\bar{x}, \bar{y}, \bar{z}$-coordinates such that the direction of the $\bar{z}$-axis is defined by the angles Θ_*, φ_* in the x, y, z-system. The averaged Lagrangian becomes in these variable coordinates somewhat simpler

$$\langle 2\Lambda\rangle = \langle \varkappa^2\overline{\Theta}_\theta^2 + \varkappa^2 \sin^2\overline{\Theta}\overline{\varphi}_\theta^2 + 2m^2\cos\overline{\Theta}\rangle - 2m^2(1+2\gamma_v)^{1/2}G(\Theta_v, \Theta_*, \varphi_v - \varphi_*) + 2\alpha\gamma_v - c_0^{-2}v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

In order to find the functions $\overline{\Theta}(\theta)$, $\overline{\varphi}(\theta)$ we must compute the stationary points in terms of $\overline{\Theta}, \overline{\varphi}$, and of the Lagrange multipliers m, Θ_*, φ_*. The stationary points in Θ_*, φ_* are easily found since these multipliers appear under the integral sign

$$\varphi_* = \varphi_v + \pi i, \quad \Theta_* = (-1)^i\Theta_v + \pi j \quad (i, j \text{ — are integers})$$

The value of the expression $G(\Theta_v, \Theta_*, \varphi_v - \varphi_*)$ at the stationary points is ± 1. We introduce the quantity

$$\gamma = \pm(1+2\gamma_v)^{1/2} - 1$$

The sign is chosen in accordance with the sign of the above expression at the stationary points, and then we can obtain the Lagrangian in the following form:

$$\langle 2\Lambda\rangle = \langle \varkappa^2\overline{\Theta}_\theta^2 + \varkappa^2\sin^2\overline{\Theta}\overline{\varphi}_\theta^2 - 2m^2(1-\cos\overline{\Theta})\rangle + 2\gamma(\alpha - m^2) + \alpha\gamma^2 - c_0^{-2}v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

The expression under the averaging sign coincides, with the accuracy of up to the notation, with the Lagrangian function for a heavy point on a smooth sphere. The sum of the first two terms of this expression is analogous to the kinetic energy, and the third one to the potential energy. The equations of motion of a spherical pendulum have to obvious first integrals, namely the energy and the surface integral

$$\varkappa^2\overline{\Theta}_\theta^2 + \varkappa^2\sin^2\overline{\Theta}\overline{\varphi}_\theta^2 + 2m^2(1-\cos\overline{\Theta}) = 4m^2K \qquad (2.2)$$
$$\varkappa\sin^2\overline{\Theta}\overline{\varphi}_\theta = 2mC$$

where K and C are the integration constants. The constant K may assume values ranging from the minimum of the expression

$$\sin^2\frac{\overline{\Theta}}{2} + \frac{C^2}{\sin^2\overline{\Theta}} \qquad (2.3)$$

in $\overline{\Theta}$, to plus infinity. If K falls within this interval, then the segment $[0,\pi]$ contains exactly two values, $\overline{\Theta} = \overline{\Theta}_1$ and $\overline{\Theta} = \overline{\Theta}_2$, where $\overline{\Theta}_\theta = 0$. In the interval between these values the difference between K and the expression (2.3) is positive, and the value of the function $\overline{\Theta}(\theta)$ oscillates periodically between $\overline{\Theta}_1$ and $\overline{\Theta}_2$.

Next we calculate the averaged Lagrangian. Using (2.2) we obtain

$$\langle 2\Lambda\rangle = \langle 2\varkappa^2\overline{\Theta}_\theta^2 + 2\varkappa^2\sin^2\overline{\Theta}\overline{\varphi}_\theta^2 - 4m^2K\rangle + 2\alpha\gamma^2 + 2\gamma(\alpha - m^2) - c_0^{-2}v_t^i v_{it} - h^2 v_{\xi\xi}^i v_{i\xi\xi} = \frac{\varkappa^2}{\pi}I_1 + \frac{\varkappa^2}{\pi}I_2 - 4m^2K + 2\gamma(\alpha - m^2) + \alpha\gamma^2 - c_0^{-2}v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi}$$

$$I_1 = \int_0^{2\pi}\overline{\Theta}_\theta^2\,d\theta, \quad I_2 = \int_0^{2\pi}\sin^2\overline{\Theta}\overline{\varphi}_\theta^2\,d\theta$$

Suppose that 2π is the smallest period of $\overline{\Theta}(\theta)$. Then the function $\overline{\Theta}$ will vary over this period from $\overline{\Theta}_1$ and $\overline{\Theta}_2$ and back. In this case then the first integral will be equal to

$$I_1 = \frac{4m}{\varkappa}\Phi(K,C), \quad \Phi(K,C) = \int_{\overline{\Theta}_1}^{\overline{\Theta}_2}\left(K - \sin^2\frac{\overline{\Theta}}{2} - \frac{C^2}{\sin^2\overline{\Theta}}\right)^{1/2} d\overline{\Theta} \qquad (2.4)$$

If on the other hand 2π is not the smallest period, then the integral will be larger by as many times, as many times 2π is larger than the smallest period, i.e.

$$I_1 = \frac{4ml}{\varkappa}\Phi(K,C)$$

where l is a natural number. The following expression follows from the surface integral:

$$\varkappa I_2 = 2mC\int_0^{2\pi}\overline{\varphi}_\theta\,d\theta = 2mC\,[\overline{\varphi}(2\pi) - \overline{\varphi}(0)]$$

Having analyzed the formulas connecting the initial functions $\psi^i(\theta)$ with the functions $\overline{\Theta}, \overline{\varphi}$, we conclude that the necessary condition for ψ^i to be periodic in θ is, that $\overline{\varphi}(2\pi) - \overline{\varphi}(0) = 2\pi n$ where n is an integer. A segment of length $2\pi/\theta_\xi$ can accommodate l periods of the function $\overline{\Theta}[\theta(\xi, t)]$ as function of ξ, at fixed t. It follows therefore that if l is large, then the wave length $2\pi/\theta_\xi$ cannot serve as the measure of the "smoothness" of the rod axis.

We shall therefore modify the initial formulation of the problem. Let the smallest period of the function $\bar{\Theta}(\theta)$ be equal not to $2\pi/l$ but to π, and assume that the functions $\psi^i(\theta)$ are 2π-periodic. The averaging sign will denote an integral in θ from 0 to πl referred to πl or, as before, an integral from 0 to 2π referred to 2π (this does not affect the value of $\langle 2\Lambda\rangle$). Then the averaged Lagrangian will become

$$\langle 2\Lambda\rangle = 8\varkappa m\pi^{-1}\Phi(K, C) + 8\varkappa mCnl^{-1} - 4m^2K + 2\gamma(\alpha - m^2) + \alpha\gamma^2 - c_0^{-2}v_t^iv_{it} + h^2v_{\xi\xi}^iv_{i\xi\xi} \tag{2.5}$$

The quantity n/l has a simple geometrical meaning in analogy with a mathematical pendulum.

If we denote the increment of $\bar{\varphi}$ corresponding to θ varying from 0 to π, i.e. over the period of the function $\bar{\theta}(\theta)$, by $\Delta\bar{\varphi}$, then we have $\Delta\bar{\varphi} = 2\pi n/l$. The inequalities /4/ $\pi < |\Delta\bar{\varphi}| < 2\pi$ hold for $\Delta\bar{\varphi}$, and this implies that $1/4 < (n/l)^2 < 1$. The cases $(n/l)^2 = 1/4$ and $(n/l)^2 = 1$ correspond to the plane oscillations of a rod and pendulum. The integers n and l appear in (2.5) in the form n/l. This allows us to consider not only the rational values of n/l, but also the real values. Moreover n/l can be regarded as a slowly varying function of ξ and t.

3. Modulation equations.

The demand that the Lagrangian (2.5) be stationary with respect to K, C and m, yields

$$\frac{8\varkappa m}{\pi}\frac{\partial\Phi}{\partial C} + 8\varkappa m\frac{n}{l} = 0,\quad \frac{8\varkappa m}{\pi}\frac{\partial\Phi}{\partial K} - 4m^2 = 0$$
$$\frac{8\varkappa}{\pi}\Phi + 8\varkappa C\frac{n}{l} - 4\gamma m - 8Km = 0 \tag{3.1}$$

Let us obtain the functions Φ, $\partial\Phi/\partial K$, $\partial\Phi/\partial C$ in the explicit form. We perform the substitution $\cos\bar{\Theta} = u$ in the integral (2.4) defining Φ, and it then assumes the form

$$\Phi = \int_b^a \frac{[1/2(a-u)(u-b)(u-c)]^{1/2}}{1-u^2}du \tag{3.2}$$

where a, b and c are the roots of the polynomial $K(1-u^2) + 1/2(1-u^2)(1-u) - C^2$. We have the following inequalities /4/:

$$c \leqslant -1 \leqslant b < a \leqslant 1;\quad a + b \geqslant 0 \tag{3.3}$$

The function Φ can be written in terms of a complete elliptic integral of the first $K(k)$ and third $\Pi(\pi/2, n, k)$ kind /5/:

$$\Phi = \sqrt{\frac{2}{a-c}}\Big[(2K - 1 + c)K(k) + (b-c)\Pi\Big(\frac{\pi}{2}, k^2, k\Big) - C^2\Big(\frac{1}{1-a}\Pi\Big(\frac{\pi}{2}, n_-, k\Big) + \frac{1}{1+a}\Pi\Big(\frac{\pi}{2}, n_+, k\Big)\Big)\Big]$$
$$k^2 = \frac{a-b}{a-c},\quad n_\pm = \frac{a-b}{a\pm 1}$$

and the functions $\partial\Phi/\partial K$ and $\partial\Phi/\partial C$ can be found more conveniently by differentiating the expression (3.2). The relations (3.1) will now become

$$m = \frac{2}{\pi}\sqrt{\frac{2}{a-c}}K(k) \tag{3.4}$$
$$\pi\frac{n}{l} = C\sqrt{\frac{2}{a-c}}\Big[\frac{1}{1-a}\Pi\Big(\frac{\pi}{2}, n_-, k\Big) + \frac{1}{1+a}\Pi\Big(\frac{\pi}{2}, n_+, k\Big)\Big]$$
$$(\gamma + 1 - c)K(k) = (b-c)\Pi\Big(\frac{\pi}{2}, k^2, k\Big)$$

The third expression in (3.4) is of geometrical character: γ is the measure of the "mean" compression of the rod, while K and C determine the amplitude of the oscillations. It is clear that in the case of an inextensible rod the above quantities must be functionally interdependent. After solving the second and third relation of (3.4) for K and C, the averaged Lagrangian will be a function of α, $\varkappa$, γ, n/l, v_t^i, $v_{\xi\xi}^i$ of the form

$$\langle 2\Lambda\rangle = 2F(\alpha, \gamma, \varkappa, n/l) - c_0^{-2}v_t^iv_{it} + h^2v_{\xi\xi}^iv_{i\xi\xi}$$

For the function $F(\alpha, \varkappa, \gamma, n/l)$ we can obtain, e.g. the following expression:

$$2F = \frac{16\varkappa^2}{\pi^2(a-c)}K^2(k)(\gamma + 1 - a - b - c) + \alpha\gamma^2 + 2\alpha\gamma \tag{3.5}$$

where a, b and c are understood to be the functions of γ and n/l. The modulation equations are obtained from the averaged variational principle

$$\delta \iint (2F - c_0^{-2} v_t^i v_{it} + h^2 v_{\xi\xi}^i v_{i\xi\xi})\, dt\, d\xi = 0$$

and the Euler equations for this principle have the form

$$c_0^{-2} v_{tt}^i = \left(\frac{v_\xi^i}{1+\gamma} \frac{\partial F}{\partial \gamma} \right)_\xi - h^2 v_{\xi\xi\xi\xi}^i \qquad (3.6)$$
$$\left(\frac{2\alpha}{\theta_t} \frac{\partial F}{\partial \alpha} \right)_t = \left(\frac{2\alpha}{\theta_\xi} \frac{\partial F}{\partial \alpha} - h \frac{\partial F}{\partial \varkappa} \right)_\xi$$

The first group of equations represents the equations for an extensible effective rod with the elastic line $x^i = v^i(\xi, t)$, provided that $(1+\gamma)^{-1}\,\partial F/\partial\gamma$ denotes the tensile force P referred to the area of transverse cross section and Young's modulus. We note that following relation:

$$P = \frac{1}{1+\gamma} \frac{\partial F}{\partial \gamma} = \alpha - \varkappa^2 A\left(\gamma, \frac{n}{l}\right), \quad A\left(\gamma, \frac{n}{l}\right) = \frac{8K^2(k)}{\pi^2 (a-c)(1+\gamma)} \qquad (3.7)$$

which can be considered as a dispersion relation connecting the tensile force, wave velocity θ_t/θ_ξ, wave length $2\pi/\theta_\xi$ and tne parameters γ and n/l determining the oscillation amplitudes.

4. Small amplitudes approximation. The expression (3.5) for the function $F(\alpha, \varkappa, \gamma, n/l)$ is written in the implicit form. To obtain the explicit expression we must solve the second and third equation of (3.4) which are transcendental. We shall therefore seek an approximate expression for this function under the assumption that the oscillation amplitudes are small. We replace K and C by k and $\lambda = (1-a)^{1/2}(1+b)^{-1/2}$. According to the inequalities (3.3) k^2 and λ^2 vary from 0 to $+1$.

Using the analogy with a mathematical pendulum we find that the rod oscillation amplitude is small if K and C are small. It can be shown that in this case the numbers k and λ are also small. Therefore the second third equation of (3.4) yield the following approximate expressions:

$$\gamma = -k^2 - 2\lambda^2 - {}^1/_8 k^4 + 3\lambda^2 k^2 \qquad (4.1)$$
$$\left(\frac{n}{l}\right)^2 = \frac{1}{4}\left[1 + \frac{k^2}{1+r} + \lambda^2(2r+1)\right], \quad r^2 = \frac{k^2+\lambda^2}{\lambda^2} \qquad (4.2)$$

At small amplitudes the quantity r has a simple geometric meaning. The trajectory of the small pendulum oscillations is approximately elliptical. The number r is equal to the ratio of the principal semiaxes of this ellipse. For the rod this means that r is equal to the ratio of the largest to the smallest amplitude of the transverse oscillations. Thus r varies from $+1$ to infinity. The case $r=1$ corresponds to the motion of a pendulum along a circumference. It can be shown that in this case the elastic line of the rod is a helix. The case of an infinite r corresponds to plane oscillations. When the amplitudes are small, then in every case the value of the measure of "average compression" γ is almost zero and the value of $(n/l)^2$ is nearly equal to 1/4.

At small amplitudes the trajectory of the mathematical pendulum is an ellipse of constantly varying orientation. It can be shown that during every "turn", i.e. on varying θ by 2π, the ellipse rotates by a small angle $\Delta = 2\pi(2|n|/l - 1)$. For a rod this means that in the case of small amplitude oscillations the elastic line in every instant of time resembles a helical line flattened in one of the directions perpendicular to the axis by r times, and twisted in such a manner that every turn is rotated, with respect to the neighboring turn, by the angle Δ. In the linear approximation the rotation is absent, since it describes a nonlinear interaction between the oscillations at different planes.

Expanding the right-hand side of (3.5) up to the fourth power in k and λ and using (4.1), we obtain the following expression for the function F:

$$2F = (-2\gamma + 4\lambda^4 + 8\lambda^2 k^2 + {}^1/_2 k^4)\varkappa^2 + \alpha\gamma^2 + 2\alpha\gamma \qquad (4.3)$$

Using now (4.1) in which only the terms quadratic in k and λ need to be retained, and from (4.2), we find the expression for k and λ in terms of γ and n/l. Substituting these expressions into (4.3), we obtain the required expression for function F:

$$2F = \varkappa^2\left[-2\gamma + \frac{\gamma^2}{2}\left(1 + \frac{3}{p^2} - 2q\right)\right] + \alpha\gamma^2 + 2\alpha\gamma \qquad (4.4)$$
$$p = -{}^3/_2\gamma\,[(4n/l)^2 - 1]^{-1}, \quad q = p^{-2}(2p^2 + 2p\sqrt{p^2-1} - 1)^{-1}$$

We note that the term $2q$ is small when the oscillations are almost plane. If $r \geqslant 3$, the value of the term does not exceed 0.08 and decreases with increasing r as r^{-4}. Therefore for such oscillations the expression (4.4) simplfies and at $(n/l)^2 = {}^1/_4$ it becomes identical to the expression for plane oscillations obtained in /2/. The expansion of the function $A(\gamma, n/l)$ in terms of small amplitudes has the form

$$A = 1 - \frac{3}{2}\gamma + \frac{105}{64}\gamma^2 - \frac{1}{16}\left[4\left(\frac{n}{l}\right)^2 - 1\right]^2 + \frac{\gamma^2}{4}q$$

When the oscillations are almost plane, the last term is small and at $r \geqslant 3$ it becomes smaller than $0.01\gamma^2$. Since $\gamma < 0$, it follows that $A > 1$ (the penultimate term is not greater than $9\gamma^2/64$ by virtue of the inequality $r \geqslant 1$). From this it follows that the greater the amplitude, the higher the wave velocity. When the amplitude tends to zero, the relation (3.7) becomes a dispersion relation for the linear transverse waves in the rod $P = \alpha - \varkappa^2$.

REFERENCES

1. SHENIAVSKII L.A., Influence of geometrical nonlinearity on the waves propagating through a thin, free plate. PMM Vol.43, No.6, 1979.
2. BERDICHEVSKII V.L., Dispersion of nonlinear waves in a rod. PMM Vol.46, No.2, 1982.
3. WHITHAM J.B., Linear and Nonlinear Waves. New York, London, Wiley, 1974.
4. APPEL' P., Theoretical Mechanics. Vol.1, Moscow, FIZMATGIZ, 1960.
5. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products. Moscow, NAUKA, 1971.

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